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Note

The Calculation of Eigenvalues for the Stationary Perturbation of Symmetrical Pipe Poiseuille Flow

1. Introduction

The eigenvalues for the stationary perturbation for plane Poiseuille flow have been presented in [1–3]. In particular [1] shows that these eigenvalues can be used in an upstream boundary condition for the two-dimensional viscous flow in a stepped channel. This paper obtains similar eigenvalues for axisymmetric perturbations of pipe Poiseuille flow. It is hoped that these eigenvalues will be used in the boundary conditions for pipe flow at some future date. The perturbations are substituted into the Navier–Stokes equations and a linear approximation taken. This yields a set of differential eigenvalue equations for the decay of a stationary perturbation very similar to those in [3] but having coefficients $1/r$ and $1/r^2$, where r is the non-dimensional distance from the centre of the pipe. The differential equations can either be solved in this form or they can be multiplied by r or r^2. Both formulations are used but, as pointed out by [4], care has to be taken with the boundary conditions if the equations contain $1/r$ and $1/r^2$ to make sure all the terms in the equations are finite at the origin.

The dependent variables of the ordinary differential equations are expressed as an expansion of Chebyshev polynomials. Two methods are used: a method using the orthogonality properties of the Chebyshev polynomials and a collocation method. The orthogonal method is analogous to that used by Orszag [5] in his treatment of the Orr–Sommerfeld stability equations and the collocation method similar to that described by Picken [6]. Both methods convert the differential eigenvalue problem into a generalized algebraic eigenvalue problem which is solved by using the QZ matrix algorithm.

Both formulations of the equations are solved using the orthogonal and collocation methods. The collocation method has the advantage that it is more straightforward to programme and that it seems to give fewer spurious eigenvalues. The eigenvalues calculated will not only be of use in the numerical solution of steady pipe flow but also give an understanding on the rate of decay of disturbances in pipes.

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2. EQUATIONS

For symmetrical pipe flow the nondimensional equations of motion are

$$\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}=0, \tag{2.1}$$

$$\nabla^2\eta-\frac{\eta}{r^2}-R\left[v_r\frac{\partial\eta}{\partial r}+v_z\frac{\partial\eta}{\partial z}-\frac{\eta v_r}{r}\right]=0, \tag{2.2}$$

$$\frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r}=\eta, \tag{2.3}$$

where

$$\nabla^2\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}$$

and v_r, v_z, η are respectively the velocity components in the radial direction and in the direction of flow and the vorticity. Cylindrical polar co-ordinates are used. The Reynolds number is given by $R=2aU/\upsilon$, where a is the radius of the pipe, U is the average velocity, and υ is the kinematic viscosity of the fluid. Equations (2.1)–(2.3) are satisfied by the pipe Poiseuille flow solution of the form

$$v_r=0, \qquad v_z=(1-r^2), \qquad \text{and} \qquad \eta=2r. \tag{2.4}$$

Following [1] we look for a perturbation solution like

$$v_r=\varepsilon V_r(r)\,e^{-\alpha z}, \qquad v=1-r^2+\varepsilon V_z(r)\,e^{-\alpha z}, \qquad \eta=2r+\varepsilon Z(r)\,e^{-\alpha z}, \tag{2.5}$$

where ε is small. Substituting (2.5) into Eqs. (2.1)–(2.3) and neglecting squares of ε leads to

$$DV_r+\frac{1}{r}V_r-\alpha V_z=0, \tag{2.6}$$

$$DV_z+\alpha V_r+Z=0, \tag{2.7}$$

$$D^2Z+\frac{D}{r}Z+\alpha^2 Z-\frac{1}{r^2}Z+R(1-r^2)\,\alpha Z=0, \tag{2.8}$$

where the operator D denotes differentiation with respect to r. The boundary conditions are

$$V_r(0)=Z(0)=V_r(1)=V_z(1)=0, \tag{2.9}$$

with the added condition that $V_z(0)$ must be finite. We will refer to Eqs. (2.6)–(2.8) as the velocity/vorticity formulation of the differential eigenvalue problem.

From [7] it is possible to deduce an alternative formulation of the problem with $V_r(z)$, $V_z(z)$, and $P(z)$ as the eigenfunctions. The equations are

$$\left(D+\frac{1}{r}\right)V_r-\alpha V_z=0, \tag{2.10}$$

$$\left[D^2+\frac{D}{r}-\frac{1}{r^2}+\alpha^2+R\alpha(1-r^2)\right]V_r-RDP=0, \tag{2.11}$$

$$\left[D^2+\frac{D}{r}+\alpha^2+R\alpha(1-r^2)\right]V_z+2rRV_r+R\alpha P=0, \tag{2.12}$$

where $P(z)$ is the eigenfunction associated with pressure. The boundary conditions are

$$V_z(0) \text{ is finite,} \qquad P(0) \text{ is finite,} \qquad V_r(0)=0, \tag{2.13}$$

and

$$V_r(1)=V_z(1)=0. \tag{2.14}$$

We will refer to Eqs. (2.10)–(2.12) as the velocity/pressure formulation of the differential eigenvalue problem.

3. Numerical Solution

Both sets of eqautions are solved in the forms given in the previous section using Chebyshev polynomials in a method first described by [5], but also described in [1] in a similar problem to the one under consideration. Let the Chebyshev expansion of $\phi(r)$ and its derivatives $d^q\phi/dr^q$ be

$$\frac{d^q\phi(r)}{dr^q}=\sum_{n=0}^{\infty} a_n^{(q)}T_n(r), \tag{3.1}$$

where $\phi(r)$ is any of the eigenfunctions $V_r(r)$, $V_z(r)$, $Z(r)$, or $P(r)$ and T_n is the nth-degree Chebyshev polynomial of the first kind defined by $T_n(\cos\theta)=\cos n\theta$, for $n=0, 1, 2, \ldots$. The Chebyshev polynomials are defined over the range $-1 \leqslant r \leqslant 1$ but the problem only uses the range $0 \leqslant r \leqslant 1$. The Chebyshev expansion method has been used in [1, 5] so little reference will be made to the details of the method. The Chebyshev expansions of the derivative and other terms listed in [1, 5] together with extra results from [4, 8] are used to convert the differential eigenvalue problem into a generalised algebraic eigenvalue problem. We use a Chebyshev series of order M.

Let us now look at the velocity/vorticity formulation in particular.

Equations (2.6)–(2.8) are very similar to the velocity/vorticity equations of plane Poiseuille flow of [3] and are solved in a similar manner. It will be noted that Eq. (2.8) seems to uncouple from (2.6) and (2.7) but it is in fact coupled through the boundary conditions. There is only one boundary condition for Z and as (2.8) is a second-order differential equation a unique solution is not possible. Reference [4] mentions the need to impose extra "pole conditions" at $r=0$ due to the terms containing $1/r$ and $1/r^2$. In the velocity/vorticity formulation these extra conditions are not required as

$$\frac{1}{r}DZ-\frac{Z}{r^2}=D\left(\frac{Z}{r}\right)$$

and with $Z(0)=0$ and appropriate differentiability conditions at $r=0$ it follows that this term remains finite as $r\to 0$. Equations (2.6)–(2.8) together with the boundary conditions (2.9) can be expressed in the form

$$(A-\alpha B)\,\mathbf{b}=0 \tag{3.2}$$

where A and B are square matrices of dimension $4M+4$ and the vector $\mathbf{b}$ contains the Chebyshev coefficients of U, V, Z, and αZ. The method used to solve (3.2) is described by [9] and is a generalisation of the standard QZ algorithm.

Let us now look at the velocity/pressure formulation. In order to reduce the number of eigenfunctions and thus the size of the matrices in the generalised eigenvalue we substitute for αV_z in Eq. (2.12) using Eq. (2.10). As suggested by [4] we must apply the extra boundary condition or "pole condition," $DV_z(0)=0$. This is required to make the term $(1/r)\,DV_z$ finite at $r=0$. Equations (2.10), (2.11), and a modified (2.12) are now solved using the Chebyshev method as in the velocity/vorticity formulation.

The detailed results obtained using the formulations described in this section are not given in detail as the method described in the next section gives similar results with less computer time. It should be noted that the velocity/pressure formulation of the equations gives extra eigenvalues just as described in [3] for the same formulation of plane Poiseuille flow. With only using half the range of the Chebyshev series there are many spurious eigenvalues and the calculations need to be performed for several M so that the correct eigenvalues can be detected.

4. Numerical Solution of the Transformed Equations

Gottlieb and Orszag [4] suggest that when using a Chebyshev series to solve ordinary differential equations with coefficients like $1/r$ and $1/r^2$ it might be better to multiply the equations by r or r^2 and avoid having to introduce an extra boundary condition at $r=0$. We also transform the range $0\leqslant r\leqslant 1$ into $-1\leqslant x\leqslant 1$, using a linear transformation. The complete range of the Chebyshev series will now be used. Equations (2.6)–(2.8) of the velocity/vorticity fomulation transform into

$$(x+1)\,V_r' + V_r - \frac{x+1}{2}\,\alpha V_z = 0, \tag{4.1}$$

$$2V_z' + Z + \alpha V_r = 0, \tag{4.2}$$

$$(x^2+2x+1)\,Z'' + (1+x)\,Z' - Z - \frac{R}{16}\,(x^4+4x^3+2x^2-4x-3)\,\alpha Z$$

$$+\frac{1}{4}\,(x^2+2x+1)\,\alpha^2 Z = 0, \tag{4.3}$$

where the prime denotes differentiation with respect to x. The boundary conditions are

$$V_r(1) = V_z(1) = 0. \tag{4.4}$$

Having multiplied the equations by r or r^2 we do not need to introduce a "pole condition." In addition the boundary conditions $V_r(-1)=0$ and $Z(-1)=0$ are satisfied by the equations themselves. The Chebyshev series method is now applied.

Equations (2.10)–(2.12) describing the problem in the velocity/pressure form are now transformed in a similar manner, and we obtain

$$(x+1)\,V_r' + V_r - \frac{x+1}{2}\,\alpha V_z = 0, \tag{4.5}$$

$$(x^2+2x+1)\,V_r'' + (x+1)\,V_r' - V_r \frac{-R\alpha}{16}\,(x^4+4x^3+2x^2-4x-3)\,V_r$$

$$+\frac{1}{4}\,(x^2+2x+1)\,\alpha^2 V_r - \frac{R}{2}\,(x^2+2x+1)\,P' = 0, \tag{4.6}$$

$$2(x+1)\,V_z'' + 2V_z' + \frac{R}{4}\,(5+2x+x^2)\,V_r + \frac{R}{8}\,(3+x-3x^2-x^3)\,V_r'$$

$$+\,(x+1)\,\alpha V_r' + \alpha V_r + R\alpha R = 0. \tag{4.7}$$

The boundary conditions are given in (4.4) and again no extra "pole conditions" are required. Equations (4.5)–(4.7) are solved using Chebyshev series method.

5. Collocation Method

Both formulations of the problem are now solved using Chebyshev polynomials in a collocation method similar to that in [6]. As before the functions are expressed in the form of finite Chebyshev series:

$$\phi(x)=\sum_{k=0}^{N} a_k T_k(x), \qquad \phi'(x)=\sum_{k=0}^{N-1} a_k^{(1)} T_k(x), \qquad \phi''(x)=\sum_{k=0}^{N-2} a_k^{(2)} T_k(x). \tag{5.1}$$

There are four eigenfunctions in each formulation so we need to obtain $4N+4$ equations. The transformed equations of Section 4 will be used. We select the $N-1$ collocation points $x_i=\cos(i\pi/N)$, $i=1, 2,..., N-1$. The boundary conditions are either

$$V_r(1)=V_z(1)=0 \quad \text{and} \quad Z(-1)=\alpha Z(-1)=V_r(-1)=0 \tag{5.2}$$

in the case of the velocity/vorticity formulation, or

$$V_r(1)=\alpha V_r(1)=V_z(1)=0 \quad \text{and} \quad V_r(-1)=\alpha V_r(-1)=0 \tag{5.3}$$

in the case of the velocity/pressure formulation. In either formulation three more equations are required and so we apply either Eqs. (4.1)–(4.2) or Eqs. (4.5)–(4.6) at $x=1$. There are now $4N+4$ equations with $4N+4$ unknowns.

We form a $(N-1)\times(N+1)$ matrix of the values of the Chebyshev polynomials at the collocation points. Reference [6] solves the problem by expressing the a_i and $a_i^{(1)}$, $i=1, 2, 3$, etc., in terms of $a_i^{(2)}$ but we will follow (5) and express the $a_i^{(2)}$ and $a_i^{(1)}$ in terms of a_i and then solve for the a_i.

The method proceeds as in the orthogonal method and we solve as in Section 3 an algebraic eigenvalue problem similar to Eq. (3.2). The amount of computer time taken using this method is much the same as the orthogonal method. The numerical results are in agreement with those of Sections 3 and 4 but the main advantage with this method is the easier programming.

6. Results

The results are presented in Tables I–III. Tables I and II give the real eigenvalues of smallest modulus. There are real eigenvalues for Reynolds number $R=10.0$ but no eigenvalues for $R\leqslant 5.0$. It should be noticed that the negative real eigenvalues tend to a constant as R becomes large while the positive real eigenvalues behave like $1/R$ as R becomes large. This behaviour is very much like the results for plane Poiseuille flow in [1]. The graphs of Reynolds number against the real eigenvalues are very similar to the equivalent graphs in [1] and for the sake of brevity are not given in this paper. The complex eigenvalues whose real parts have the smallest modulus are given in Table III. The complex eigenvalues with negative real parts are not presented for Reynolds number $R\geqslant 100$ because due to the large negative real part they are less accurate. Reference [1] explains that eigenvalues with positive real parts are associated with downstream disturbances while eigenvalues with negative real parts are associated with upstream disturbances. Then for

TABLE I

Real Positive Eigenvalues

Reynolds number R	Eigenvalue α		
10	2.84882	4.10718	5.28426
25	1.23820	5.90935	
50	0.63541	1.86985	5.49656
100	0.320002	0.950323	1.89301
250	0.128266	0.382302	0.763792
500	0.0641523	0.191314	0.382500
1000	0.0320785	0.0956774	0.191328
2000	0.0160396	0.0478413	0.0956735

Reynolds numbers greater than 10 both the upstream and downstream perturbations from pipe Poiseuille flow are dominated by the real eigenvalue of smallest modulus. It will be noticed that in Table III for Reynolds number $R = 10$ the complex eigenvalue with positive real part seems out of line with the other entries. This type of behaviour is explained in [2]. At $R = 10$ the complex eigenvalue with real part near 3.7 becomes two real eigenvalues. The complex eigenvalue presented here is really the eigenvalue with next largest modulus.

The orthogonal method of Orszag does throw up spurious eigenvalues on occasions and care has to be exercised, but the collocation method does not give these spurious eigenvalues. The boundary conditions are less of a problem when the equations are multiplied by r or r^2. The computational time is very much the same for both the orthogonal method and the collocation method but the collocation method is far easier to programme.

TABLE II

Real Negative Eigenvalues

Reynolds number R	Eigenvalue α		
10	-4.94606		
25	-4.49291	-8.13980	-12.4276
50	-4.31816	-7.77838	-11.2475
100	-4.19980	-7.57708	-10.9217
250	-4.09219	-7.40731	-10.6829
500	-4.03427	-7.31888	-10.5649
1000	-3.99010	-7.25228	-10.4778
2000	-3.95604	-7.20123	-10.4116

TABLE III

Complex Eigenvalues

Reynolds number R	Complex eigenvalue α with: Negative real part		Positive real part	
0.25	-4.51264	± 1.47383	4.42107	± 1.45948
0.5	-4.56012	± 1.47848	4.37695	± 1.44994
1.0	-4.65849	± 1.48232	4.29200	± 1.42646
2.5	-4.98237	± 1.44170	4.06288	± 1.32452
5.0	-5.64817	± 1.07722	3.76255	± 1.06197
10	-8.31614	± 1.53700	7.85193	± 0.681963
25	-18.6746	± 1.4003	3.7072	± 0.5094
50	-19.5699	± 3.3245	3.7920	± 0.4524
100			4.17041	± 0.272606
250			3.80091	± 0.0625023
500			4.00175	± 0.160450
1000			3.87390	± 0.16162
2000			3.9263	$\pm 8.52 \times 10^{-2}$

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